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MIXED CONVECTION IN AN AXISYMMETRIC BUOYANT PLUME

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Abstract—The mixed convection in an axisymmetric point heat source due to a vertical free stream is analyzed using the boundary layer equations. The two solutions where the buoyancy effects are favourable and adverse with respect to the oncoming stream are studied. An analytical argument indicates that the solutions for the favourable case are unique whereas those for the adverse case are dual. The numerical solutions for the favourable case have been obtained for all flow regimes from pure forced convection to pure free convection buoyant plume. For the adverse case the flow in the plume is retarded and the dual numerical solutions correspond to the direct and reverse flows in the plume.

NOMENCLATURE

- c_p, specific heat at constant pressure;
- $E_1(t)$, exponential integral, defined by equation (40);
- f, nondimensional stream function, defined by equation (15);
- F, nondimensional stream function, defined by equation (8);
- h, nondimensional temperature, defined by equation (15);
- H, nondimensional temperature, defined by equation (8);
- Q, rate of heat release from point source;
- T, temperature;
- T_∞, free stream temperature;
- u, axial velocity component;
- v, normal velocity component;
- U_∞, free stream velocity;
- x, axial coordinate in the vertical direction;
- y, normal coordinate.

Greek symbols

- α, mixed convection parameter, defined by equation (14);
- $\hat{\alpha}$, mixed convection parameter, defined by equation (B6);
- β, $\alpha^{-1/2}$;
- $\hat{\beta}$, defined by equation (B3);
- ξ, χ, optimal coordinates, defined by equation (22);
- ζ, η, similarity variables, defined by equations (7) and (15);
- θ_p, reference temperature, defined by equation (8);
- ε, an index, $\varepsilon = 1$ for the favourable case, and $\varepsilon = -1$ for the adverse case;
- ν, kinematic viscosity;
- ρ, fluid density;
- σ, Prandtl number of fluid;
- ψ, stream function.

INTRODUCTION

THE MIXED convection in an axisymmetric buoyant plume is of interest in several engineering applications such as hot wire anemometry and the dispersion of pollutants. The weakly buoyant plumes have been studied by Wesseling [1]. Using the Oseen–Boussinesq linearization of the Navier–Stokes equations, the solution to the first perturbation for the velocity has been obtained. The estimation of further higher order perturbations in the scheme [1] is extremely difficult. If the buoyancy and forced convection effects are comparable, the Oseen linearization fails and one has to consider the full non-linear equations. In the present work the entire non-linear mixed convection regime has been investigated when the free stream is vertical using the boundary layer equations, for the two situations where buoyancy effects accelerate and retard the flow in the plume.

The effects of buoyancy in the mixed convection are qualitatively similar to pressure gradients in the boundary layer. When the buoyant force vector assists the main stream, the flow in the plume is accelerating and the situation corresponds to that of a favourable pressure gradients in a boundary layer flow. When the buoyant force vector opposes the main stream, the flow in the plume is retarded and the situation corresponds to that of an adverse pressure gradient in a boundary layer flow.

The mixed convection problem due to the point heat source in a vertical free stream admits self-similarity unlike the 2-dim. heat source [2]. The limiting case of purely free convection buoyant plume has been studied [3, 4]. The numerical solutions to non-linear self-similar equations have been obtained for air. For the favourable case, the solutions have been obtained for the entire non-linear mixed convection regime ranging from pure forced to free convection flows. In the adverse case the numerical solutions are dual corresponding to the direct and reverse flows in the plume.

2. SIMILARITY ANALYSIS

The boundary layer equations for an axisymmetric fluid motion under the Boussinesq approximation are

$$y\frac{\partial u}{\partial x} + \frac{\partial (yv)}{\partial y} = 0, \quad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\varepsilon g\beta(T-T_\infty)+\frac{v}{y}\frac{\partial}{\partial y}\left(y\frac{\partial u}{\partial y}\right), \quad (2)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\frac{v}{\sigma y}\frac{\partial}{\partial y}\left(y\frac{\partial T}{\partial y}\right). \quad (3)$$

Here $\varepsilon = 1$ for the favourable case where buoyancy accelerates the flow in the plume and $\varepsilon = -1$ for the adverse case where buoyancy decelerates the flow in the plume. The velocity and temperature profiles are symmetric with respect to the vertical axis and far away approach their free stream values,

$$y = 0, \quad v = \frac{\partial u}{\partial y} = \frac{\partial T}{\partial y} = 0, \quad (4)$$

$$y \to \infty, \quad u \to U_\infty, \quad T \to T_\infty. \quad (5)$$

The integration of the energy equation (3) yields

$$\rho c_p \int_0^\infty u(T-T_\infty)y\,\mathrm{d}y = \frac{Q}{2\pi} \quad (6)$$

where Q is the heat released from the point source.

When the buoyancy effects modifies the forced convection flow, it is expected that the flow will behave like an axisymmetric wake. Thus we consider the following variables:

$$\zeta = \frac{U_\infty}{2vx}y^2, \quad \theta_p = \frac{Q}{2\pi\rho c_p v x}, \quad (7)$$

$$\psi = vxF(\zeta), \quad T-T_\infty = \theta_p H(\zeta). \quad (8)$$

Introducing the variables (7) and (8) in the governing equations (1)–(6) we get

$$(\zeta F'')'+\tfrac{1}{2}FF''+\varepsilon\alpha H = 0. \quad (9)$$

$$\sigma^{-1}(\zeta H')'+\tfrac{1}{2}(FH)' = 0. \quad (10)$$

The boundary conditions are

$$\zeta = 0,\ F = \zeta^{1/2}F'' = \zeta^{1/2}H' = 0, \quad (11)$$

$$\zeta \to 0, \quad F' \to 1, \quad H \to 0, \quad (12a,b)$$

subject to the normalized heat flux condition

$$\int_0^\infty F'H\,\mathrm{d}\zeta = 1. \quad (13)$$

Here α, a parameter, defined by

$$\alpha = \frac{Gr_x}{2R_x^2} = \frac{g\beta Q}{4\pi\rho c_p v U_\infty^2} \quad (14)$$

is a measure of the relative importance of the buoyancy effect with respect to forced convection. The existence of a solution with respect α is given in Appendix A.

For the favourable case, when $\alpha \to \infty$ the buoyancy force becomes dominant and the situation may be studied by introducing the following variables:

$$\eta = \zeta/\beta, \quad f(\eta) = F(\zeta), \quad h(\eta) = H(\zeta). \quad (15)$$

The condition that at large values of α all the terms in momentum equation (9) are of equal order leads to

$$\beta = \alpha^{-1/2}. \quad (16)$$

In the new variables (15), equations (9)–(13) reduce to

$$(\eta f'')'+\tfrac{1}{2}ff''+h = 0, \quad (17)$$

$$(\eta h')'+(\sigma/2)(fh)' = 0, \quad (18)$$

$$\eta = 0, \quad f = \eta^{1/2}f'' = \eta^{1/2}h' = 0, \quad (19a,b,c)$$

$$\eta \to \infty, \quad f' \to \beta, \quad h \to 0. \quad (20a,b)$$

$$\int_0^\infty f'h\,\mathrm{d}\eta = 1. \quad (21)$$

For $\beta = 0$ the above equations reduce to those of an axisymmetric free convection buoyant plume [3, 4]. From transformation (7), it may be anticipated that the paraboloidal coordinates are optimal in the sense of Kaplun [5]. The optimal coordinates (ξ, χ) can easily be worked out to be

$$x = \tfrac{1}{2}(\xi-\chi), \quad y = (\xi\chi)^{1/2}. \quad (22)$$

The stream function (8) may be written in terms of optimal coordinates as

$$\psi = \frac{v\xi}{2}F\left(\frac{U_\infty\chi}{v}\right) \quad (23)$$

$$\psi \sim \frac{U_\infty}{2}(\xi\chi - v\xi\Delta), \quad \frac{U_\infty\chi}{v} \to \infty \quad (24)$$

where Δ is given by

$$\Delta = \lim_{\zeta\to\infty}(\zeta - F) = \lim_{\eta\to\infty}(\beta\eta - f). \quad (25)$$

The second term in equation (24) is due to the displacement effects in the outer layer invisid flow associated with the viscous buoyant plume and shows that the stream surfaces of the displacement flow are paraboloids of revolution corresponding to a constant value of ξ. The solution (23) to boundary layer flow is valid everywhere, to order v, in the flow field as it contains the outer displacement solution as a special case when $U_\infty\chi/v \to \infty$.

3. PERTURBATION EXPANSIONS

For the weakly buoyant plumes, the variables F and H in equations (9) and (10) are expanded in terms of α as

$$F = \sum_{n=0} F_n(\zeta)(\varepsilon\alpha)^n, \quad (26a)$$

$$H = \sum_{n=0} H_n(\zeta)(\varepsilon\alpha)^n. \quad (26b)$$

The solution to the leading momentum F_0 is dual. One of the solutions is

$$F_0(\zeta) = \zeta \quad (27a)$$

and the corresponding solution to the energy equation is

$$H_0 = (\sigma/2)\exp(-\sigma\zeta/2). \quad (27b)$$

The second solution to the momentum problem F_0 is that of a mixing layer with a shift in origin [2, 7].

In view of equation (27), the expansions (26) are the classical Oseen linearization. The higher order terms in the expansion are given by the recurrence relations

$$(\zeta F_n'')' + \frac{\zeta}{2} F_n' = -H_{n-1} - \frac{1}{2}\sum_{r=1}^{n-1} F_r F_{n-r}'', \quad (28)$$

$$H_n' + \frac{\sigma}{2} H_n = -\frac{\sigma}{2\zeta}\sum_{r=1}^{n} F_r H_{n-r}. \quad (29)$$

The boundary and integral conditions are

$$\zeta = 0, \quad F_n = \zeta^{1/2} F_n'' = \zeta^{1/2} H_n' = 0, \quad (30a,b,c)$$

$$\zeta \to \infty, \quad F_n', \quad H_n \to 0, \quad (31a,b)$$

$$\sum_0^\infty \int_0^\infty H_n \, d\zeta = -\sum_{r=1}^{n} \int_0^\infty F_r' H_{n-r} \, d\zeta. \quad (32)$$

The solution to the first perturbation problem for $\sigma \neq 1$ is

$$F_1 = \frac{2\sigma}{\sigma - 1}\left\{\frac{\zeta}{2}[E_1(\zeta/2) - E_1(\sigma\zeta/2)] + 1 - e^{-\zeta/2} - \frac{1}{\sigma}(1 - e^{-\sigma\zeta/2})\right\}, \quad (33)$$

$$H_1 = \frac{\sigma^3}{2(\sigma-1)}\left[\frac{1}{\sigma}\ln\frac{(1+\sigma)^\sigma}{2\sigma} + \frac{1}{\sigma} \times (\sigma\zeta/2 + 1)E_1(\sigma\zeta/2) - (\zeta/2 + 1)E_1(\zeta/2) + \frac{1}{\sigma}(1 - e^{-\sigma\zeta/2}) - 1 + e^{-\zeta/2} - \frac{\sigma - 1}{\sigma} \times (\gamma + \ln\zeta - \ln 2) + \frac{1}{\sigma}\ln\sigma\right] e^{-\zeta/2} \quad (34)$$

with

$$\int_0^\infty H_1 \, d\zeta = -\frac{\sigma}{\sigma - 1}\ln\frac{\sigma + 1}{2} \quad (35)$$

and for $\sigma = 1$ is

$$F_1 = 2(1 - e^{-\zeta/2}), \quad (36)$$

$$H_1 = \tfrac{1}{2}[2\ln 2 - \tfrac{1}{2} - \gamma - \ln\zeta - E_1(\zeta/2)]e^{-\zeta/2}, \quad (37)$$

$$F_2 = (3 + 2\gamma - 2\ln 2)e^{-\zeta/2} + 2[E_1(\zeta/2) + \ln\zeta - \ln 2]e^{-\zeta/2} - e^{-\zeta} + (\zeta - 2) \times [E_1(\zeta) - E_1(\zeta/2)] - 2. \quad (38)$$

with

$$\int_0^\infty H_1 \, d\zeta = -\tfrac{1}{2}. \quad (39)$$

Here $E_1(\zeta)$ is the exponential integral, defined as

$$E_1(\zeta) = \int_\zeta^\infty e^{-t}/t \, dt \sim -\ln\zeta - \gamma - O(\zeta), \quad \zeta \to 0 \quad (40)$$

and $\gamma = 0.577$ is Eulers constant.

4. RESULTS AND DISCUSSION

The self-similar equations described in Section 2 constitute a fifth order singular non-linear boundary value problem with two missing conditions at the axis. The boundary conditions to be guessed can be reduced by one through a transformation described in Appendix B. The remaining missing boundary condition is obtained by satisfying the boundary condition at infinity in the least square sense. Further, on account of singularity in the equations at $y = 0$ the numerical integration cannot be started from there. The remedy generally employed in the literature is to find a series solution valid in the neighbourhood of $y = 0$ and then start the integration from some finite, but very small, values of y instead of $y = 0$. In the present work, however, the highest order derivatives in the equations have been estimated at $y = 0$ and using these derivatives the integration is directly started from the axis $y = 0$. The numerical integration has been carried out by the Runge–Kutta method with the Gill improvement on an IBM 1130 computer for a Prandtl number $\sigma = 0.72$.

We shall first describe our solutions for the favourable case. For computational convenience the entire range of α between zero and infinity is divided into two subranges. For the subrange $0 \leqslant \alpha \lesssim 1$ the solutions are computed from equations (9)–(13) and for the second subrange $1 \lesssim \alpha \lesssim \infty$ $(0 \leqslant \beta \lesssim 1)$ from equations (17)–(21) with a changeover at $\alpha = \beta = 1$. The velocity and temperature profiles for the first subrange $0 \leqslant \alpha \lesssim 1$ are displayed in Fig. 1 and for second subrange $1 \leqslant \alpha \leqslant \infty$ are displayed in Fig. 2. The various characteristics of the buoyancy layer are given in Tables 1 and 2. The velocity and temperature at the axis and the entrainment of the fluid in the plume are displayed in Fig. 3 (see also Tables 1 and 2). The results for the first subrange $(0 \leqslant \alpha \leqslant 1)$ are displayed linearly with α while for the second subrange $(1 \leqslant \alpha \leqslant \infty)$ they are shown linearly with $1/\alpha$ $(=\beta^2)$. Thus Fig. 3 covers the entire regime for the favourable case from pure forced convection $(\alpha = 0)$ to pure free convection $(1/\alpha = \beta^2 = 0)$ flows. Figure 3 shows that the velocity in the plume is always greater than the free stream velocity. Further, as buoyancy effects increase the velocity in the plume also increases implying that flow is accelerating. The temperature at the axis also increased with buoyancy effects. When buoyancy effects tend to become dominant $(\alpha \to \infty)$ the temperature at the axis and the entrainment, approaches to a finite limit whereas the velocity at the axis becomes very large (like $\alpha^{1/2}$) and flow approaches to an axisymmetric buoyant plume [3, 4].

For the adverse case the buoyant force vector opposes the main stream and the flow in the plume is retarded. The numerical solutions for the velocity $F'(0)$ in the plane of symmetry and entrainment function Δ are shown in Fig. 4(a) against the mixed convection parameter α. The figure shows that for $\alpha = 0.6076$ (later denoted by α_s) the velocity at the axis $F'(0)$ vanishes. For $\alpha < \alpha_s$ the solutions are dual (see also Appendix A) corresponding to a forward velocity $[F'(0) > 0]$ and a reverse velocity $[F'(0) < 0]$ in the plume. The closed form solution obtained from Oseen-linearization in Section 5, also displayed in the Fig. 4(a), is valid only for $\alpha < 0.2$. As α increases the Oseen-

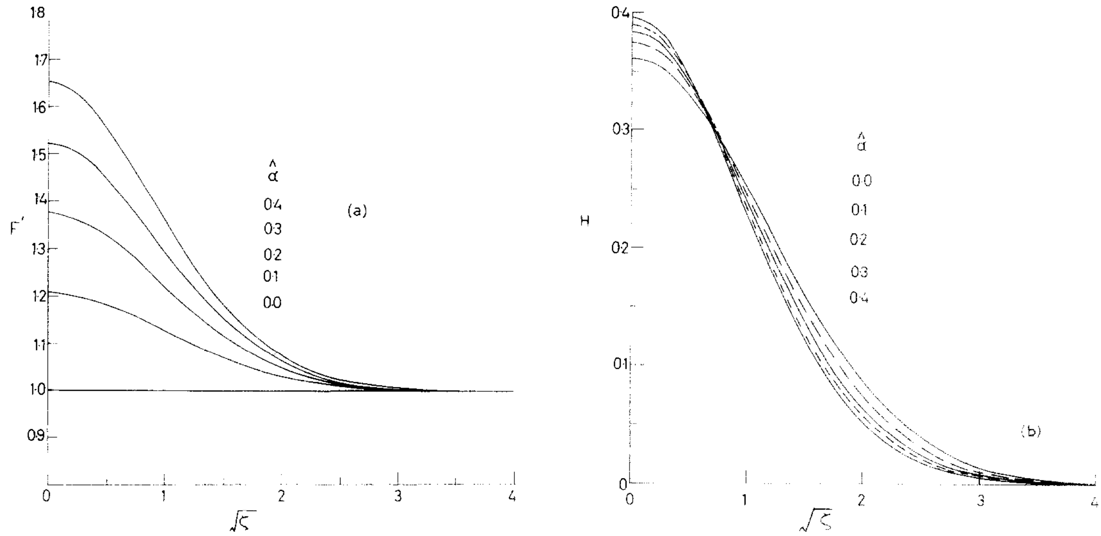

FIG. 1. Favourable case: Profiles for mixed convection buoyant plume in the range spanning from pure forced convection to non-linear mixed convection flow ($0 \leqslant \alpha \leqslant 1$). (a) Velocity profiles (b) Temperature profiles. Legend same as given in Table 1.

linearization fails and one has to study the full non-linear equations, which shows that $F'(0)$ decreases as α approaches to α_s. At $\alpha = \alpha_s$ the velocity at the axis $F'(0) = 0$ and $dF'(0)/d\alpha$ is infinite. Around $\alpha = \alpha_s$ the curve turns back with the result that as α decreases from its value α_s velocity at the axis $F'(0)$ decreases to become negative. The fact that $F'(0) = 0$ for $\alpha = 0$ and α_s, in the reverse flow case, suggests that $F'(0)$ should be minimum at $\alpha = \alpha_m$, i.e. the magnitude of reverse flow velocity is maximum. The solutions for the entrainment function Δ displayed in Fig. 4(a) also clearly shows the duality of the solutions. For a given α the two solutions for Δ are positive and the larger value corresponds to the reverse flow case. This is physically reasonable as in the reverse flow case the thickness of the plume is much more than the forward flow case and a larger amount of fluid is entrained. The temperature at the axis of the plume $H(0)$ is displayed in Fig. 4(b) against α. Here again the dual solutions for $\alpha < \alpha_s$ predict a positive temperature at the axis. In the forward flow domain when $\alpha = 0$, $H(0) = \sigma/2$, as α increases the value of $H(0)$ decreases and at $\alpha = \alpha_s$, $dH(0)/d\alpha$ is infinite. In the

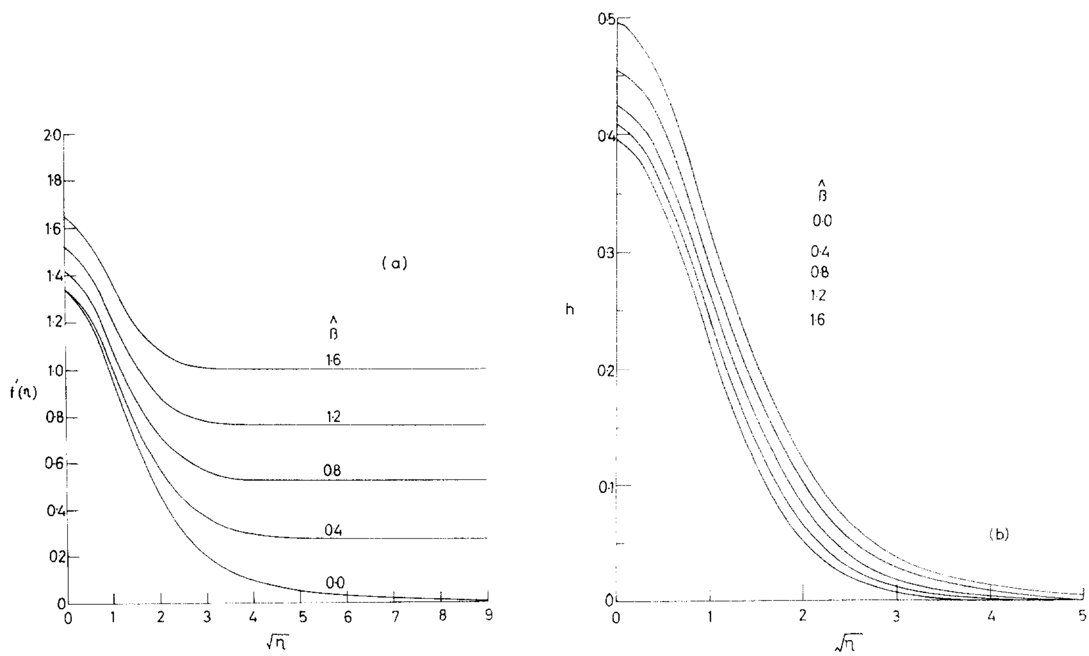

FIG. 2. Favourable case: Profiles for mixed convection buoyant plume in the range spanning from non-linear mixed convection to convection buoyant plume ($1 \leqslant \alpha \leqslant \infty$; $\beta = \alpha^{-1/2}$). (a) Velocity profile. (b) Temperature profiles. Legend same as given in Table 2.

Table 1. Favourable case: Characteristic values for the first subrange ($0 \leqslant \alpha \leqslant 1$) for mixed convection buoyant plume

$\bar{\alpha}$	α	$F'(0)$	$H(0)$	$-\Delta$
0	0	1.0	0.36	0.0
0.1	0.2677	1.2059	0.3735	0.441
0.2	0.5226	1.3744	0.3827	0.7541
0.3	0.7698	1.5207	0.3897	0.9934
0.4	1.0018	1.6519	0.3653	1.1888

reverse flow domain as α decreases from α_s to zero the temperature $H(0)$ decreases to zero continuously. The numerical results for the thickness of boundary layer δ (defined as value of y where $u = 0.99U_\infty$) the thickness of reverse flow layer δ_t and various other characteristics are given in Table 3.

For the adverse case the numerical results for profiles of velocity $F'(\zeta)$ and temperature $H(\zeta)$ are displayed in Figs. 5 and 6, respectively, and the curves labelled $1, 2, 3, \ldots, 11$ correspond to the various values of α given in Table 3. The curve labelled 6 is for the marginal case $\alpha = \alpha_s$, the curves 1–5 are for the forward flow case and the curves 6–11 are for the reverse flow case. The velocity profiles clearly display the duality of the solutions. Figure 5 shows the extent of reverse flow in the transverse direction δ_t is zero at $\alpha = \alpha_s$ (curve 6) and increases as α decreases from α_s. In the limit as $\alpha \to 0$, the thickness of reverse flow layer δ_t increases but the magnitude of the velocity $F'(0)$ at the axis approaches zero. The temperature profiles displayed in Fig. 6 show that for the forward flow case (curves 1–6) the temperature monotonically decreases from its value $H(0)$, at the axis, to zero at infinity. However, in the reverse flow case (curves 7–11) the temperature profiles show a maximum (say, $H = H_{max}$ at $\zeta = \zeta_{max}$). As α decreases from its value α_s the ζ_{max} increases. H_{max} decreases continuously and $\zeta_{max} \to \infty$ as $H_{max} \to 0$. The above behaviour of the velocity and temperature profiles suggests that as $\alpha \to 0^-$ the velocity profile approaches a mixing layer, similar to that of Chapman, with a shift in origin [6]. The above mixing layer arises in a manner analogous to that of Kennedy and Stewartson (see Chapter 4 of ref. [6]) where the pressure gradient in a self-similar incompressible wake approaches zero.

Table 2. Favourable case: Characteristic values for the second subrange ($1 \geqslant \beta \geqslant 0$) for mixed convection buoyant plume

$\bar{\beta}$	β	$f'(0)$	$h(0)$	$-\Delta$
0	0	1.3339	0.4963	6.6506
0.4	0.2695	1.3489	0.4538	3.5747
0.5	0.5219	1.4140	0.4256	2.3149
1.2	0.7657	1.5171	0.4071	1.6079
1.6	1.0054	1.6490	0.3948	1.1715

In the reverse flow situation, where the thickness of the boundary layer (in terms of transformed coordinate, say, ζ_∞) is appreciably larger than in the forward flow situation, the utility of the boundary layer equations may appear somewhat limited. If y_∞ is the thickness of boundary layer in the physical coordinate then from equation (7) we have

$$(y_\infty/x)^2 = \zeta_\infty 2\nu/U_\infty x. \quad (41)$$

The boundary layer equations are valid for $y_\infty \ll x$ and the condition (41) is satisfied for

$$x \gg \nu\zeta_\infty/U_\infty. \quad (42)$$

The condition (41) implies that for small values of ζ_∞ the validity of the solutions extends to the near source region whereas for large values of ζ_∞, the solution is

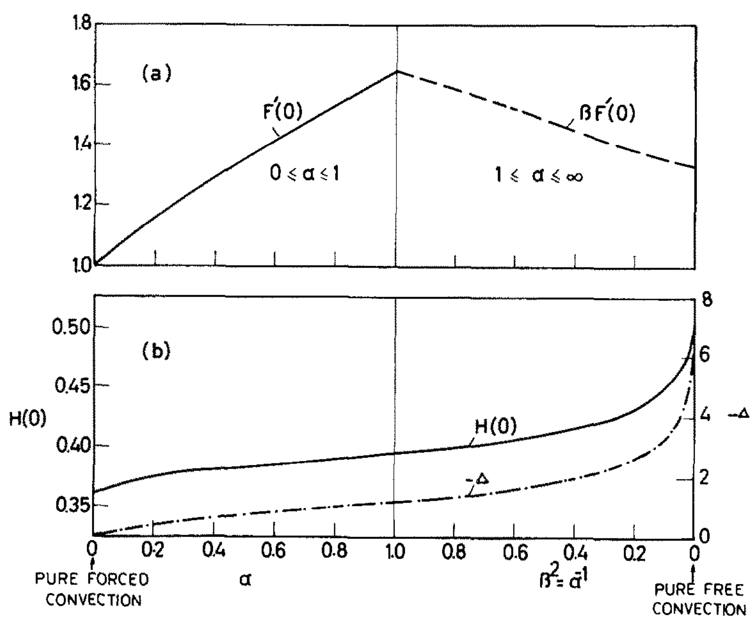

FIG. 3. Favourable case: The velocity and temperature in the plane of symmetry and entrainment function Δ against the mixed convection parameter α.

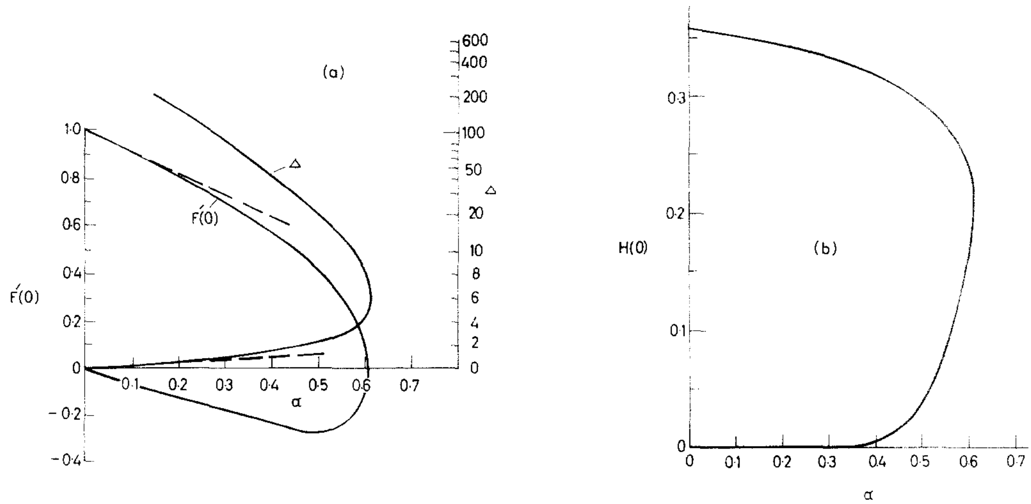

FIG. 4. Adverse case: The flow characteristics of the mixed convection buoyant plume. (a) Non-dimensional velocity $F'(0)$ at the axis and entrainment parameter Δ. (b) Non-dimensional temperature $H(0)$ at the axis against mixed convection parameter α. The dotted lines show the results from Oseen-linearization.

valid relatively far from the heat source. In the neighbourhood of the region $x \sim \zeta_\infty \nu / U_\infty$ the boundary layer equations fail and one has to consider the full Navier–Stokes equations.

5. CONCLUSIONS

(a) For the favourable case, where buoyancy effects accelerate the flow in the plume, the solution to the mixed convection problem is unique.

(b) For the adverse case, the buoyancy retards the flow in the plume. For $\alpha < \alpha_s$ the solutions are dual leading to forward and reverse flows in the plume. For $\alpha = \alpha_s$ the solution is unique, corresponding to the marginal state between the forward and reverse flows. For $\alpha > \alpha_s$ the solution to the problem does not exist.

(c) The nature of dual solutions in the adverse case is summarized below: For a given

(i) $\alpha < \alpha_s$, the solutions are dual;

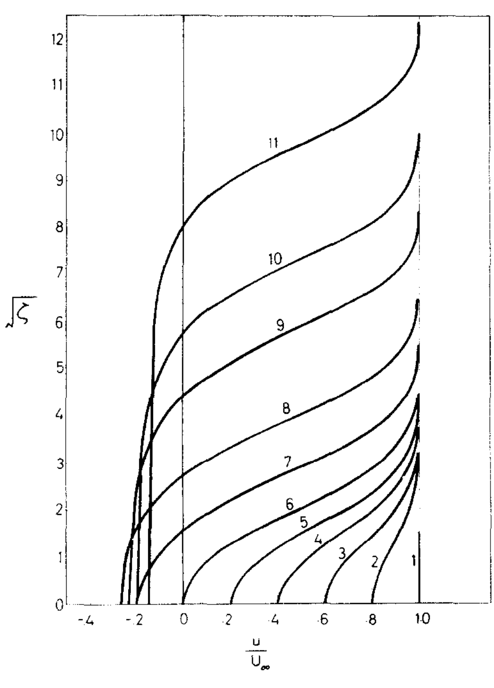

FIG. 5. Adverse case: Velocity profiles for the mixed convection buoyant plumes. Curves labelled 1, 2, 3, ..., 11 correspond to values of α given in Table 3.

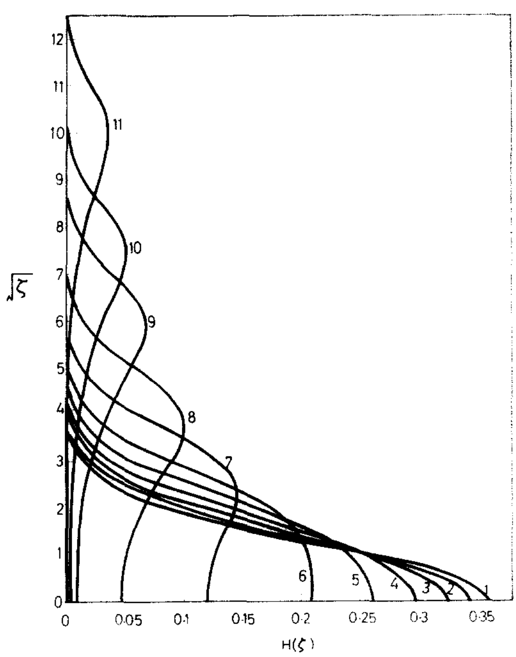

FIG. 6. Adverse case: Temperature profiles for the mixed convection buoyant plume. Curves labelled 1, 2, 3, ..., 11 correspond to the values of α given in Table 3.

Table 3. Adverse case: Characteristic values of buoyancy layer for various values of mixed convection parameter α

Legend in Figs. 5 and 6	$F'(0)$	α	$H(0)$	$\delta[U_\infty/(2\nu x)]^{1/2}$	Δ	$\delta_t[U_\infty/(2\nu x)]^{1/2}$
1	1.0	0.0	0.36	0	0	
2	0.8	0.2141	0.3438	2.7823	0.5230	
3	0.6	0.3826	0.3234	3.2209	1.1817	
4	0.4	0.5052	0.2969	3.8932	2.0479	
5	0.2	0.5811	0.2608	4.0908	3.2695	
6	0.0	0.6076	0.2083	4.3204	5.2594	0.0
7	−0.2	0.5728	0.1188	5.0889	10.0744	1.633
8	−0.27	0.5112	0.4650E-1	6.1613	18.9371	2.750
9	−0.24	0.4054	0.9818E-2	7.7822	39.1690	4.485
10	−0.20	0.3609	0.3146E-2	9.3969	60.1973	5.820
11	−0.15	0.2675	0.5765E-3	12.1201	105.0362	8.116

(ii) $F'(0) > 0$, the solution is unique;
(iii) $F'(0) < 0$, the solution for $H(0)$ is unique whereas the values of α are dual;
(iv) $H(0)$, the solution is unique.

(d) In the adverse case the Oseen-linearization can be fruitfully employed to study the forward flow situation for small values of α (<0.2). In the reverse flow situation the solution for $\alpha \to 0$ approaches a mixing layer with a shift in origin.

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APPENDIX A

EXISTENCE OF SOLUTIONS

The existence of the solution to the equations (9)–(13) in a weaker sense can be analyzed by employing the Oseen-linearization, where F is replaced by ζ in non linear terms. The integration of equations (9) and (10) under boundary conditions (11) and (12) yields

$$F' = 1 - \frac{2\varepsilon A\alpha}{\sigma - 1}[E_1(\sigma\zeta/2) - E_1(\zeta/2)],$$

$$H_1 = A\exp(-\sigma\zeta/2), \quad A > 0.$$

Using the above solution the heat flux relation (13) yields a quadratic equation

$$\varepsilon\alpha A^2 + \alpha_s(-2A + \sigma) = 0, \qquad \text{(A1)}$$

where

$$\alpha_s = (\sigma - 1)\Big/\left[4\sigma \ln\left(\frac{1+\sigma}{2}\right)\right]. \qquad \text{(A2)}$$

The two roots of the equation (A1) are

$$A = \frac{1 \pm (1 + \varepsilon\alpha/\alpha_s)^{1/2}}{-\varepsilon\alpha/\alpha_s}. \qquad \text{(A3)}$$

For the adverse case ($\varepsilon = -1$) the two roots are distinctly positive and equal provided $\alpha \lesssim \alpha_s$. For $\sigma = 0.72$, equation (A2) predicts $\alpha_s = 0.608$. The slight error in the two values of α_s is due to the fact that the linearization employed here is not valid around $\alpha \simeq \alpha_s$. The roots (A3) are complex for $\alpha > \alpha_s$ implying the non-existence of the solutions.

For the favourable case ($\varepsilon = 1$), only one of the two roots in equation (A3) is positive, implying that the unique solution exists for all values of σ and α.

APPENDIX B

METHOD OF SOLUTION

The self-similar equations in Section 2 form a non-linear two-point boundary value problem. The two missing conditions to be guessed can be reduced to one by the transformation given below.

The equations (17), (18) and (20b) are invariant under the transformation

$$f(\eta) = \hat{f}(\hat{\eta}), \quad h(\eta) = \hat{h}(\hat{\eta})/J, \quad \eta = \hat{\eta}/J \qquad \text{(B1)}$$

but the conditions (21a) and (22) are changed to

$$\partial\hat{f}/\partial\hat{\eta} = \beta(J)^{1/2} = \hat{\beta} \quad \text{(say)}, \int_0^\infty \hat{f}'(\hat{\eta})\hat{h}\,d\hat{\eta} = J \qquad \text{(B2)}$$

where J is a constant. For a given $\hat{\beta}$, we can fix $\hat{h}(0) = 1$, and guess $\hat{f}'(0)$ such that the boundary condition $\hat{f}'(\infty) = \hat{\beta}$ is satisfied. The converged solution $(\hat{f}, \hat{h})$ will lead to the value J defined by equation (B2). The converged solution can be transformed in terms of original variables by equation (B1) and the values of β is given by

$$\beta = \hat{\beta}/J^{1/2}. \qquad \text{(B3)}$$

In another case equations (9)–(13) are invariant under the transformation

$$\zeta = \hat{\zeta}, \quad F(\zeta) = \hat{F}(\hat{\zeta}), \quad \alpha H(\zeta) = \hat{\alpha}H(\hat{\zeta}) \qquad \text{(B4)}$$

but the integral condition is changed to

$$\int_0^{\infty} \hat{F}'\hat{H}\, d\hat{\xi} = I. \tag{B5}$$

For a given $\hat{\alpha}$, we can again fix $\hat{H}(0) = 1$, and guess $\hat{F}'(0)$ such that the boundary condition at infinity is satisfied. The converged solution is transformed in terms of original variables (B3) and the values of α is given by

$$\alpha = \hat{\alpha}/I. \tag{B6}$$

Although without loss of generality we can fix $\hat{H}(0) = 1$, for computing dual solutions in the adverse case near $\alpha \simeq 0$ it has been found convenient to fix $H(0)$ at relatively smaller values.

CONVECTION MIXTE DANS UN PANACHE AXISYMETRIQUE

Résumé—La convection mixte pour une source ponctuelle de chaleur et un écoulement libre vertical est analysée en employant les équations de la couche limite. Les deux solutions pour les effets de gravitation favorables ou adverses sont étudiées en rapport avec l'écoulement incident. Un argument analytique indique que les solutions pour le cas favorable ont été obtenues pour tout le régime d'écoulement allant de la convection forcée pure à la convection naturelle pure. Dans le cas adverse, l'écoulement dans le panache est retardé et les solutions duales numériques correspondent aux écoulements direct et de retour dans le panache.

GEMISCHTE KONVEKTION IN EINER ACHSENSYMMETRISCHEN AUFTRIEBSSTRÖMUNG

Zusammenfassung—Die gemischte Konvektion an einer achsensymmetrischen punktförmigen Wärmequelle, die durch eine vertikale freie Strömung bestimmt ist, wird mit Hilfe der Grenzschichtgleichungen untersucht. Es werden die beiden Fälle betrachtet, in denen die Auftriebseffekte die vorhandene Grundströmung verstärken oder ihr entgegenwirken. Die analytische Behandlung zeigt, daß die Lösungen für den verstärkenden Fall eindeutig sind, während sich für den entgegenwirkenden Fall doppeldeutige Lösungen ergeben. Die numerischen Lösungen für den verstärkenden Fall wurden für den gesamten Bereich der Strömungszustände von reiner erzwungener Konvektion bis zu reiner freier Konvektion der Auftriebsströmung berechnet. Im entgegengerichteten Fall wird die Strömung im Auftriebsgebiet verzögert, und die doppeldeutigen numerischen Lösungen entsprechen der direkten und der Rückströmung im Auftriebsgebiet.

СМЕШАННАЯ КОНВЕКЦИЯ В ОСЕСИММЕТРИЧНОЙ ПЛАВУЧЕЙ СТРУЕ

Аннотация—Смешанная конвекция в случае осесимметричного точечного источника тепла, вызываемая вертикальным свободным течением, проанализирована с помощью уравнений пограничного слоя. Изучены два решения, когда сила плавучести имеет такое же направление, как натекающий поток, или противоположна ему. Анализ указывает на единственность решения для первого случая и наличие двух решений для второго. Численные решения для первого случая получены во всем диапазоне режимов течения, начиная с чисто вынужденной конвекции и кончая свободноконвективной струей. Во втором случае течение в струе замедлено, а два решения соответствуют прямому и обратному течениям в струе.